

Some operator convex functions of several variables

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Abstract

We obtain operator concavity (convexity) of some functions of two or three variables by using perspectives of regular operator mappings of one or several variables. As an application, we obtain, for $0 < p < 1$, concavity, respectively convexity, of the Frechét differential mapping associated with the functions $t \rightarrow t^{1+p}$ and $t \rightarrow t^{1-p}$.

1 Introduction and preliminaries

We study convexity or concavity of certain operator mappings. Some of them may be expressed by the functional calculus for functions of several variables while others are of a more general nature.

1.1 The functional calculus

Let \mathcal{H} denote an n -dimensional Hilbert space. The space $B(\mathcal{H})$ of bounded linear operators on \mathcal{H} is itself a Hilbert space with inner product given by $(A, B) = \text{Tr}(B^*A)$ for $A, B \in B(\mathcal{H})$.

Definition 1.1. Let $f: I_1 \times \cdots \times I_k \rightarrow \mathbb{R}$ be a function defined in a product of real intervals, and let X_1, \dots, X_k be commuting operators on H_n with spectra $\sigma(X_i) \subseteq I_i$ for $i = 1, \dots, k$. We say that the k -tuple (X_1, \dots, X_k) is in the

domain of f . Consider the spectral resolution

$$X_m = \sum_{i_m=1}^{n_m} \lambda_{i_m}(m) P_{i_m}$$

where $\lambda_1(m), \dots, \lambda_{n_m}(m)$ for $m = 1, \dots, k$ are the eigenvalues of X_m . The functional calculus is defined by setting

$$f(X_1, \dots, X_k) = \sum_{i_1=1}^{n_1} \cdots \sum_{i_k=1}^{n_k} f(\lambda_{i_1}(1), \dots, \lambda_{i_k}(k)) P_{i_1}(1) \cdots P_{i_k}(k)$$

which makes sense since $\lambda_{i_m}(m) \in I_m$ for $i_m = 1, \dots, n_m$ and $m = 1, \dots, k$.

Since the operators X_1, \dots, X_k in the above definition are commuting all of the spectral projections $P_{i_m}(m)$ do also commute. The functional calculus therefore defines $f(X_1, \dots, X_k)$ as a self-adjoint operator on \mathcal{H} . Notice that if the tuples (X_1, \dots, X_k) and (Y_1, \dots, Y_k) are in the domain of f then so is the tuple $(\lambda X_1 + (1 - \lambda)Y_1, \dots, \lambda X_k + (1 - \lambda)Y_k)$ for $\lambda \in [0, 1]$.

In order to study convexity properties of the functional calculus it is convenient to consider commuting C^* -subalgebras $\mathcal{A}_1, \dots, \mathcal{A}_k$ of $B(\mathcal{H})$ and require that $X_m \in \mathcal{A}_m$ for $m = 1, \dots, k$. For more details on the functional calculus the reader may refer to [12, 7, 8].

The restriction of the functional calculus by f to k -tuples of operators $(X_1, \dots, X_k) \in \mathcal{A}_1 \times \cdots \times \mathcal{A}_k$ in the domain of f is said to be convex if

$$\begin{aligned} f(\lambda X_1 + (1 - \lambda)Y_1, \dots, \lambda X_k + (1 - \lambda)Y_k) \\ \leq \lambda f(X_1, \dots, X_k) + (1 - \lambda)f(Y_1, \dots, Y_k) \end{aligned}$$

for $\lambda \in [0, 1]$.

Definition 1.2. Let $f: I_1 \times \cdots \times I_k \rightarrow \mathbb{R}$ be a function defined in a product of real intervals. We say that f is matrix convex of order n if the restriction of the functional calculus by f to operators $(X_1, \dots, X_k) \in \mathcal{A}_1 \times \cdots \times \mathcal{A}_k$ in the domain of f is convex for arbitrary commuting C^* -subalgebras $\mathcal{A}_1, \dots, \mathcal{A}_k$ of $B(\mathcal{H})$.

1.2 More general operator mappings

However, not all mappings defined on operators can be expressed in the form of the functional calculus by some function. This is especially common for mappings of several variables.

Hansen introduced the notion of regular operator mappings of several variables [11] based on earlier investigations of regular mappings of two variables [6, 5]. Furthermore, Hansen introduced the notion of the perspective of a regular operator mapping of several variables in continuation of earlier results obtained for functions of one variable by Effros [4], see also [3]. As an application of these ideas we obtain convexity (concavity) statements for some three-variable operator mappings. As a corollary we are able to prove that some concrete functions of three variables are operator convex.

We also prove operator concavity (convexity) of the Frechét differential mapping associated with the power functions $t \rightarrow t^p$ for $p \in (0, 2]$. Hansen [10] and Chen and Tropp [2] proved independently that the inverse Frechét differential associated with the operator monotone functions $t \rightarrow t^p$ is a concave mapping in positive definite operators, where $0 < p \leq 1$. In the present paper we investigate similar problems for the operator convex functions $t \rightarrow t^{1+p}$ and obtain that the associated Frechét differential mapping is concave in positive definite operators.

2 Operator convex functions of two variables

We begin by studying some operator concave (convex) functions of two variables in order to derive concavity (convexity) of the Frechét differential mapping associated with the power functions.

Theorem 2.1. *Let $0 < p \leq 1$. The two-variable function*

$$G(s, t) = \begin{cases} \frac{t^{p+1} - s^{p+1}}{t - s} & t \neq s \\ \frac{1}{p+1} t^p & t = s, \end{cases}$$

defined in $(0, \infty) \times (0, \infty)$, may be extended to a concave map defined in pairs of positive definite operators in $B(\mathcal{H})$. In particular, it is operator concave.

Proof. Consider for each $\lambda \in [0, 1]$ the mapping

$$f_\lambda(A, B) = (\lambda A + (1 - \lambda)B)^p$$

defined in pairs of positive definite operators acting on \mathcal{H} . Consider furthermore, for $\alpha \in [0, 1]$, convex combinations $A = \alpha A_1 + (1 - \alpha)A_2$ and

$B = \alpha B_1 + (1 - \alpha)B_2$ of pairs of positive definite operators (A_1, A_2) and (B_1, B_2) , then

$$\begin{aligned}
f_\lambda(A, B) &= (\lambda A + (1 - \lambda)B)^p \\
&= [\alpha(\lambda A_1 + (1 - \lambda)B_1) + (1 - \alpha)(\lambda A_2 + (1 - \lambda)B_2)]^p \\
&\geq \alpha(\lambda A_1 + (1 - \lambda)B_1)^p + (1 - \alpha)(\lambda A_2 + (1 - \lambda)B_2)^p \\
&= \alpha f_\lambda(A_1, B_1) + (1 - \alpha)f_\lambda(A_2, B_2),
\end{aligned}$$

where we used that $(A, B) \rightarrow \lambda A + (1 - \lambda)B$ is affine and $A \rightarrow A^p$ is concave in positive definite operators. Therefore, $(A, B) \rightarrow f_\lambda(A, B)$ is concave. Since, for $s \neq t$, the integral

$$\begin{aligned}
\int_0^1 (\lambda t + (1 - \lambda)s)^p d\lambda &= \frac{1}{p+1} \int_0^1 \frac{d}{d\lambda} \left(\frac{(\lambda t + (1 - \lambda)s)^{p+1}}{t - s} \right) d\lambda \\
&= \frac{1}{p+1} \frac{t^{p+1} - s^{p+1}}{t - s},
\end{aligned}$$

we obtain by continuity that G may be extended to a concave operator mapping defined in positive definite operators. **QED**

The following result is by method related to [10, Theorem 4.1].

Theorem 2.2. *Consider for $0 < p \leq 1$ the function $f(t) = t^{p+1}$ defined in the positive half-line. The Frechét differential mapping $A \rightarrow df(A)$ is concave in positive definite operators.*

Proof. Let A be a positive definite operator diagonalized with respect to a basis $\{e_i\}_{i=1}^n$ such that $Ae_i = \lambda_i e_i$ for $i = 1, \dots, n$. For any matrix $H = (h_{ij})_{i,j=1}^n$ in \mathcal{H} we then have

$$df(A)(H) = H \circ L_f(\lambda_1, \dots, \lambda_n)$$

expressed as the Hadamard product of H and the Löwner matrix

$$L_f(\lambda_1, \dots, \lambda_n) = \left(\frac{\lambda_i^{p+1} - \lambda_j^{p+1}}{\lambda_i - \lambda_j} \right)_{i,j=1}^n.$$

Hence we obtain

$$\begin{aligned}
\operatorname{Tr} H^* df(A) H &= \sum_{i,j=1}^n |h_{ij}|^2 \frac{\lambda_i^{p+1} - \lambda_j^{p+1}}{\lambda_i - \lambda_j} \\
&= \sum_{i,j=1}^n |(He_j, e_i)|^2 \frac{\lambda_i^{p+1} - \lambda_j^{p+1}}{\lambda_i - \lambda_j} \\
&= \operatorname{Tr} H^* G(L_A, R_A) H,
\end{aligned}$$

where L_A and R_A are the commuting left and right multiplication operators with respect to A . Applying the concavity of $(A, B) \rightarrow G(A, B)$ above and Theorem 1.1 in [9] we obtain that the map

$$A \rightarrow \operatorname{Tr} H^* G(L_A, R_A) H$$

is concave for any operator H acting on \mathcal{H} . The operator mapping $A \rightarrow df(A)$ is therefore concave. Notice that we only needed to invoke operator concavity of the real function $G(t, s)$. **QED**

Corollary 2.3. *Take $0 < p \leq 1$. The two-variable function*

$$F(s, t) = \begin{cases} \frac{t - s}{t^{p+1} - s^{p+1}} & t \neq s \\ \frac{1}{p+1} t^{-p} & t = s, \end{cases}$$

defined in $(0, \infty) \times (0, \infty)$, may be extended to a convex map defined in positive definite invertible operators in $B(\mathcal{H})$. In particular, it is operator convex.

Proof. Since inversion is convex and decreasing in positive definite invertible operators the result follows from Theorem 2.1. **QED**

Corollary 2.4. *Take $0 < p \leq 1$ and consider the function $f(t) = t^{p+1}$. The map $A \rightarrow df(A)^{-1}$ is then convex in positive definite matrices.*

Proof. With the same assumptions as in the proof of Theorem 2.2, the inverse Frechét differential may be expressed as the Hadamard product

$$df(A)^{-1}(H) = H \circ \left(\frac{\lambda_i - \lambda_j}{\lambda_i^{p+1} - \lambda_j^{p+1}} \right)_{i,j=1}^n,$$

hence

$$\begin{aligned}\mathrm{Tr} H^* df(A)^{-1} H &= \sum_{i,j=1}^n |(He_j, e_i)|^2 \frac{\lambda_i - \lambda_j}{\lambda_i^{p+1} - \lambda_j^{p+1}} \\ &= \mathrm{Tr} H^* F(L_A, R_A) H,\end{aligned}$$

where L_A and R_A are the left and right multiplication operators with respect to A . The statement now follows since F is operator convex. **QED**

Theorem 2.5. *Take $0 \leq p < 1$ and consider the function $f(t) = t^p$. The map $A \rightarrow df(A)$ is then convex in positive definite matrices.*

Proof. The idea is quite similar to the construction above. We consider the function

$$H(s, t) = \begin{cases} \frac{t^{1-p} - s^{1-p}}{t - s} & t \neq s \\ \frac{1}{1-p} t^{-p} & t = s \end{cases}$$

defined in $(0, \infty) \times (0, \infty)$. Since when $s \neq t$ we may write

$$H(s, t) = (1-p) \int_0^1 (\lambda t + (1-\lambda)s)^{-p} d\lambda$$

and the map

$$(A, B) \rightarrow (\lambda A + (1-\lambda)B)^{-p}$$

is convex in pairs of positive definite operators for $\lambda \in [0, 1]$, we obtain that H is operator convex. The statement now follows in the same way as in the proof of Theorem 2.2. **QED**

The obtained results may be compared with the concavity statement [10, 2] of the inverse of the Frechét differential mapping associated with the functions $t \rightarrow t^p$ for $0 < p < 1$.

3 Operator convex functions of three variables

Recently, Hansen defined the perspective of a regular operator mapping of several variables [11]. In this section, by applying the notion of perspectives, we exhibit some operator concave (convex) functions of three variables.

Consider for each $k = 1, 2, \dots$ the domain

$$\mathcal{D}_+^k = \{(A_1, \dots, A_k) | A_1, \dots, A_k > 0\}.$$

of k -tuples of positive definite invertible operators A_1, \dots, A_k acting on a Hilbert space \mathcal{H} .

Definition 3.1. Let $F : \mathcal{D}_+^k \rightarrow B(\mathcal{H})$ be a regular mapping. The perspective \mathcal{P}_F of F is the mapping defined in the domain \mathcal{D}_+^{k+1} by setting

$$\mathcal{P}_F(A_1, \dots, A_k, B) = B^{1/2} F(B^{-1/2} A_1 B^{-1/2}, \dots, B^{-1/2} A_k B^{-1/2}) B^{1/2}.$$

Hansen [11] proved the following convexity theorem.

Theorem 3.2. Let \mathcal{H} be an infinite dimensional Hilbert space. The perspective \mathcal{P}_F of a convex regular map $F : \mathcal{D}_+^k \rightarrow B(\mathcal{H})$ is convex and regular.

We use the above convexity result to obtain:

Theorem 3.3. Let $0 < p \leq 1$. The three-variable function

$$F_3(t_1, t_2, t_3) = \begin{cases} \frac{t_1^{p+1} - t_2^{p+1}}{t_1 - t_2} t_3^{1-p} & t_1 \neq t_2 \\ \frac{1}{p+1} t_1^p t_3^{1-p} & t_1 = t_2, \end{cases}$$

defined in $(0, \infty) \times (0, \infty) \times (0, \infty)$, may be extended to a concave map in positive definite invertible operators. In particular, it is operator concave.

Proof. The regular map,

$$(A, B) \rightarrow G(A, B) = (\lambda A + (1 - \lambda)B)^p,$$

is for $0 < p < 1$ concave in positive definite operators. The perspective mapping

$$\mathcal{P}_G(A, B, C) = C^{1/2} (\lambda C^{-1/2} A C^{-1/2} + (1 - \lambda) C^{-1/2} B C^{-1/2})^p C^{1/2}$$

of three variables is thus concave. In particular, the function

$$(t_1, t_2, t_3) \rightarrow (\lambda t_1 + (1 - \lambda)t_2)^p t_3^{1-p}$$

is operator concave. Since we may write

$$F_3(t_1, t_2, t_3) = (p+1) \int_0^1 (\lambda t_1 + (1 - \lambda)t_2)^p t_3^{1-p} d\lambda$$

the statement follows. **QED**

Notice that by setting $t_1 = t_2$ in the above theorem, we recover Lieb's concavity theorem [13, Theorem 1]. The same general idea gives an additional result.

Theorem 3.4. *Let $0 < p < 1$. The three-variable function*

$$F_3(t_1, t_2, t_3) = \begin{cases} \frac{t_1^{1-p} - t_2^{1-p}}{t_1 - t_2} t_3^{1+p} & t_1 \neq t_2 \\ \frac{1}{1-p} t_1^{-p} t_3^{1+p} & t_1 = t_2, \end{cases}$$

defined in $(0, \infty) \times (0, \infty) \times (0, \infty)$, may be extended to a convex map in positive definite invertible operators. In particular, it is operator convex.

Proof. The proof follows the same method as in the above theorem by first noticing that the map

$$(A, B) \rightarrow (\lambda A + (1 - \lambda)B)^{-p}$$

is convex in positive definite operators for each $\lambda \in [0, 1]$. **QED**

Theorem 3.5. *Let $0 < p \leq 1$. The function of three variables*

$$F_3(t_1, t_2, t_3) = \begin{cases} \frac{t_1 - t_2}{t_1^p - t_2^p} t_3^p & t_1 \neq t_2 \\ \frac{1}{p} t_3^p t_1^{1-p} & t_1 = t_2, \end{cases}$$

defined in $(0, \infty) \times (0, \infty) \times (0, \infty)$, may be extended to a concave mapping in positive definite operators acting on a Hilbert space. The function F_3 is in particular operator concave.

Proof. For each $0 < p \leq 1$ we consider the regular operator mapping

$$A \rightarrow F_1(A) = (\lambda A^p + (1 - \lambda)I)^{1/p}$$

defined in positive definite operators acting on a Hilbert space. We obtain that F_1 is concave by using that the function

$$t \rightarrow (\lambda t^p + (1 - \lambda))^{1/p}, \quad 0 < p \leq 1$$

is operator concave, see [1, 10]. The perspective,

$$\mathcal{P}_{F_1}(A, B) = B^{1/2}(\lambda(B^{-1/2}AB^{-1/2})^p + (1 - \lambda)I)^{1/p}B^{1/2},$$

is thus concave in positive definite invertible operators. Since the function $t \rightarrow t^{1-p}$ is operator monotone and operator concave for $0 < p \leq 1$, it follows that the regular mapping

$$G_2(A, B) = \mathcal{P}_{F_1}^{1-p}(A, B)$$

is concave in positive definite invertible operators. In particular, the integral

$$F_2(A, B) := \frac{1}{p} \int_0^1 G_2(A, B) d\lambda$$

is concave. Taking the perspective once more we obtain that

$$\begin{aligned} \mathcal{P}_{F_2}(A, B, C) &= C^{1/2}F_2(C^{-1/2}AC^{-1/2}, C^{-1/2}BC^{-1/2})C^{1/2} \\ &= \frac{1}{p}C^{1/2} \int_0^1 \left\{ (C^{-1/2}BC^{-1/2})^{1/2} \left[\lambda((C^{-1/2}BC^{-1/2})^{-1/2}C^{-1/2}AC^{-1/2} \times \right. \right. \\ &\quad \left. \left. (C^{-1/2}BC^{-1/2})^{-1/2})^p + (1 - \lambda)I \right]^{1/p} (C^{-1/2}BC^{-1/2})^{1/2} \right\}^{1-p} d\lambda C^{1/2} \end{aligned}$$

is concave in positive definite invertible operators. Since for positive numbers,

$$\mathcal{P}_{F_2}(t_1, t_2, t_3) = F_3(t_1, t_2, t_3),$$

the statements follow. **QED**

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